

The Logarithmic Fib-Binomial Formula

A.K.Kwaśniewski

Higher School of Mathematics and Applied Informatics
PL-15-021 Białystok, ul. Kamienna 17, POLAND
e-mail: kwandr@uwb.edu.pl

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Abstract

Steven Roman's Logarithmic Binomial Formula analogue has been found and is presented here also for the case of fibonomial coefficients - which recently have been given a combinatorial interpretation by the present author.

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1 Introduction

The aim of this note is to find out - as in [14, 15] of Steven Roman - the form of "Fib-corresponding" Logarithmic Fib-Binomial Formula. In [14, 15] Steven Roman introduced The Logarithmic Binomial Formula :

$$\lambda_n^{(t)}(x+a) = \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix} \lambda_{n-k}^{(t)}(a)x^k; \quad t = 0, 1; \quad |x| < a; \quad n \in \mathbf{Z}$$

where

$$[n] = \begin{cases} n & n \neq 0 \\ 1 & n = 0 \end{cases}$$

and the Roman factorial is given by

$$[n]! = \begin{cases} n! & n \geq 0 \\ \frac{(-1)^{n+1}}{(-n-1)!} & n < 0 \end{cases} \quad (1)$$

while hybrid binomial coefficients [10] (Roman coefficients [11]) read:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} \quad (2)$$

One may show that (Propositions 3.2 , 4.1, 4.2, 4.3 in [11])

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \binom{n}{k} & n, k \geq 0 \\ (-1)^k \binom{-n-1+k}{k} & k \geq 0 \geq n \\ (-1)^{k+n} \binom{-k-1}{n-k} & 0 > n \geq k \\ (-1)^{(n+k)} \left[\Delta^n \frac{1}{x-k} \right]_{k=0} & k > n \geq 0 \end{cases} \quad (3)$$

$$\begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ -k \end{bmatrix} = \frac{(-1)^{k+1}}{k}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}, \quad \begin{bmatrix} n \\ j \end{bmatrix} \cdot \begin{bmatrix} j \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} \cdot \begin{bmatrix} n-j \\ j-k \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} \quad (5)$$

As seen from the above the hybrid binomial coefficients (Roman coefficients) are the intrinsic natural extension of binomial coefficients .

The Logarithmic Binomial Formula extends the notion of binomiality of polynomials as used in the Generalized Umbral Calculus (see Chapter 6 in [16] for functional formulation and see [8] for abundant references on Finite Operator Calculus of Rota formulation).

The great invention of Steven Roman - among others - relies on the fact that the real i.e. R -linear span L of the basis functions (harmonic logarithms-see: Proposition 4.1 in [14])

$$L = \text{span} \left| \left\{ \lambda_n^{(t)} \right\}_{n \in \mathbf{Z}, t=0,1} \right|$$

allows the Fundamental Theorem of Calculus to hold on L i.e. $D^{-1}D = DD^{-1} = id_L$. Here D^{-1} depending on whether $t = 0$ or $t = 1$ acts as follows:

$$D^{-1} = \int_0^x \quad \text{on} \quad \lambda_n^{(0)} \quad n \neq -1, \quad n \in \mathbf{Z} \quad \text{and gives } 0 \text{ for } n = -1$$

$$\text{and } D^{-1} = \int_1^x \quad \text{on} \quad \lambda_n^{(1)}; \quad n \in \mathbf{Z}$$

2 Fibonomial Coefficients

In [5] Fibonomial coefficients [12, 2, 3, 4] have been given a combinatorial interpretation as counting the number of finite "birth-selfsimilar" subposets of an infinite poset. We shall use here the following notation: *Fibonomial coefficients* are defined as $\binom{n}{k}_F = \frac{F_n!}{F_k!F_{n-k}!}$ or - usefully for our purpose here $\binom{n}{k}_F \equiv \frac{n_F^k}{k_F!}$ where we make an analogy driven [8] identifications: $(n > 0)$, $n_F \equiv F_n \neq 0, n_F! \equiv n_F(n-1)_F(n-2)_F(n-3)_F \dots 2_F 1_F$; $0_F! = 1$; $n_F^k = n_F(n-1)_F \dots (n-k+1)_F$. This is the appropriate specification of notation from [8] for the purpose Fibonomial Finite Operator Calculus case investigation (see Example 2.1 in [9]).

Let us now introduce an infinite poset P (for further details see: [5]) via its finite part subposet P_m Hasse diagram to be continued ad infinitum in an obvious way as seen from the figure below. It looks like the Fibonacci tree with a specific "cobweb": see Figure 1. One sees that the P_m is the subposet of P consisting of points up to m -th level points

$$\bigcup_{s=1}^m \Phi_s ; \Phi_s \text{ is the set of elements of the } s\text{-th level}$$

How many P_m 's rooted at the k -th level might be found ?

We answer this question in the following sequence of observations right after Figure 1.

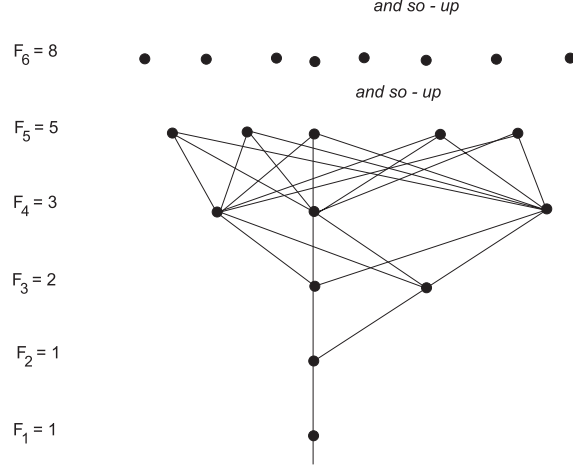


Fig. 1. The construction of the Fibonacci "cobweb" poset

Observation 2.1. *The number of maximal chains starting from the root (level F_1) to reach any point at the n -th level labeled by F_n is equal to $n_F!$*

Observation 2.2. *The number of maximal chains starting from the level labeled by F_k to reach any point at the n -th level labeled by F_n is equal to $n_F^{\frac{m}{F}}$, ($n = k + m$).*

Observation 2.3. *Let $n = k + m$. The number of subposets P_m rooted at the level labeled by F_k and ending at the n -th level labeled by F_n is equal to*

$$\binom{n}{m}_F = \binom{n}{k}_F = \frac{n_F^k}{k_F!}.$$

3 The Logarithmic Fibonomial Case

We shall now adopt the $*_\psi$ product formalism [9] (see also Appendix in [6] and [7]) to the Fibonomial case with $\exp_F \{x\} = \sum_{k=0}^{\infty} \frac{x^k}{k_F!}$ defining the F -exponential series.

3.1 $*_F$ product

Let $n > 0$ and let ∂_F be a linear operator acting on formal series and defined accordingly by $\partial_F x^n = n_F x^{n-1}$; $n \geq 0$, $\partial_F x^0 = 0$.

We shall call the F -multiplication the new $*_F$ product of functions or formal series specified below.

Notation 3.1. $x *_F x^n = \hat{x}_F(x^n) = \frac{(n+1)}{(n+1)_F} x^{n+1}$; $n \geq 0$ hence $x *_F 1 = x$ and $x *_F \alpha 1 = \alpha 1 *_F x = x *_F \alpha = \alpha *_F x = \alpha x$; $\forall x, \alpha \in \mathbf{R}$, $f(x) *_F x^n = f(\hat{x}_F) x^n$.
For $k \neq n$ $x^n *_F x^k \neq x^k *_F x^n$ as well as $x^n *_F x^k \neq x^{n+k}$ - in general.

Definition 3.1. With Notation 3.1 adopted define the $*_F$ powers of x according to

$$x^{n*}_F \equiv x *_F x^{(n-1)*}_F = \hat{x} \left(x^{(n-1)*}_F \right) = x *_F x *_F \dots *_F x = \frac{n!}{n_F!} x^n; \quad n \geq 0.$$

Note that $x^{n*}_F *_F x^{k*}_F = \frac{n!}{n_F!} x^{(n+k)*}_F \neq x^{k*}_F *_F x^{n*}_F = \frac{k!}{k_F!} x^{(n+k)*}_F$ for $k \neq n$ and $x^{0*}_F = 1$.

This noncommutative F -product $*_F$ is devised so as to ensure the observations below.

Observation 3.1.

- (a) $\partial_F x^{n*}_F = n x^{(n-1)*}_F$; $n \geq 0$;
- (b) $\exp_F[\alpha x] \equiv \exp \{ \alpha \hat{x}_F \} \mathbf{1}$;
- (c) $\exp[\alpha x] *_F \{ \exp_F \{ \beta \hat{x}_F \} \mathbf{1} \} = \exp_F \{ [\alpha + \beta] \hat{x}_F \} \mathbf{1}$;
- (d) $\partial_F (x^k *_F x^{n*}_F) = (D x^k) *_F x^{n*}_F + x^k *_F (\partial_F x^{n*}_F)$;
- (e) *Leibniz rule* $\partial_F (f *_F g) = (D f) *_F g + f *_F (\partial_F g)$; f, g - formal series;
- (f) $f(\hat{x}_F) g(\hat{x}_F) \mathbf{1} = f(x) *_F \tilde{g}$; $\tilde{g}(x) = g(\hat{x}_F) \mathbf{1}$.

3.2 F -Integration

Let $\partial_0 x^n = x^{n-1}$. The linear operator ∂_0 is identical with divided difference operator. Let $\hat{Q}f(x) = f(qx)$. Recall that to the Jackson ∂_q derivative [8] there corresponds the q -integration which is a right inverse operation to "q-difference-ization". Namely [8]

$$F(z) := \left(\int_q \varphi \right) (z) := (1-q)z \sum_{k=0}^{\infty} \varphi(q^k z) q^k \quad (6)$$

$$F(z) \equiv \left(\int_q \varphi \right) (z) = (1-q)z \left(\sum_{k=0}^{\infty} q^k \hat{Q}^k \varphi \right) (z) = \left((1-q) \hat{z} \frac{1}{1-q\hat{Q}} \varphi \right) (z) \quad (7)$$

where $(\hat{z}\varphi)(z) = z\varphi(z)$.

Of course

$$\partial_q \circ \int_q = id \quad (8)$$

as

$$\frac{1-q\hat{Q}}{1-q} \partial_0 \left((1-q) \hat{z} \frac{1}{1-q\hat{Q}} \varphi \right) = id \quad (9)$$

Naturally (9) might serve to define a right inverse to Jackson's "q-difference-ization" $(\partial_q \varphi)(x) = \frac{1-q\hat{Q}}{1-q} \partial_0 \varphi(x)$ and consequently the "q-integration" as represented by (6) and (7). As it is well known the definite q-integral is an numerical approximation of the definite integral obtained in the $q \rightarrow 1$ limit.

Finally we introduce the analogous representation for ∂_F difference-ization

$$\partial_F = \hat{n}_F \partial_0; \quad \hat{n}_F x^{n-1} = n_F x^{n-1}; \quad n \geq 1. \quad (10)$$

Then

$$\int_F x^n = \left(\hat{x} \frac{1}{\hat{n}_F} \right) x^n = \frac{1}{(n+1)_F} x^{n+1}; \quad n \geq 0 \quad (11)$$

and of course

$$\partial_F \circ \int_F = id. \quad (12)$$

Naturally $(\int_F \equiv \int d_F)$

$$\partial_F \int_a^x f(t) d_F t = f(x).$$

The formula of "per partes" F -integration is easily obtainable from Observation (3.1) and it reads:

$$\int_a^x (f *_F \partial_F g)(t) d_F t = [(f *_F g)(t)]_a^x - \int_a^x (Df *_F g)(t) d_F t. \quad (13)$$

Now in order to have ∂_F^{-1} - an F -analogue of D^{-1} as in [13, 14] (thus causing the fundamental Theorem of Calculus to hold for ∂_F -difference-ization and \int_F - integration on some linear space L_F being the linear span of

"*F-harmonic logarithms*") - we shall proceed exactly as Steven Roman in [14, 15].

4 The Logarithmic Fib-Binomial Formula

As in [14, 15] of Roman - we have also *The Logarithmic Fib-Binomial Formula* (see: Propositions 4.1, 4.2 below):

$$\phi_n^{(t)}(x+_Fa) \equiv [\exp \{a\partial_F\} \phi_n^{(t)}](x) = \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_F \phi_{n-k}^{(t)}(a)x^k \quad t = 0, 1; \quad |x| < a; \quad n \in \mathbf{Z}$$

where (more on " $+_F$ " see [8, 9])

$$[n_F] = \begin{cases} n_F & n \neq 0 \\ 1 & n = 0 \end{cases}$$

and the Roman Fib-factorial is given by

$$[n_F]! = \begin{cases} n_F! & n \geq 0 \\ \frac{(-1)^{n+1}}{(-n-1)_F!} & n < 0 \end{cases} \quad (14)$$

while *Fib-hybrid binomial coefficients* or Roman Fib-coefficients (see: [10, 11]) read:

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{[n]!}{[k]![n-k]!} \quad (15)$$

One observes (as in Propositions 3.2 , 4.1, 4.2, 4.3 in [11]) that:

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \begin{cases} \binom{n}{k}_F & n, k \geq 0 \\ (-1)^k \binom{-n-1+k}{k}_F & k \geq 0 > n \\ (-1)^{k+n} \binom{-k-1}{n-k}_F & 0 > n \geq k \\ (-1)^{(n+k)} \left[\Delta_F^n \frac{1}{x-k} \right]_{k=0} & k > n \geq 0 \end{cases} \quad (16)$$

$$\begin{bmatrix} 0 \\ k \end{bmatrix}_F = \begin{bmatrix} 0 \\ -k \end{bmatrix}_F = \frac{(-1)^{k+1}}{k_F}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \begin{bmatrix} n \\ n-k \end{bmatrix}_F, \quad \begin{bmatrix} n \\ j \end{bmatrix}_F \cdot \begin{bmatrix} j \\ k \end{bmatrix}_F = \begin{bmatrix} n \\ k \end{bmatrix}_F \cdot \begin{bmatrix} n-j \\ j-k \end{bmatrix}_F \quad (17)$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F + \begin{bmatrix} n-1 \\ k \end{bmatrix}_F \quad (18)$$

where (see: pp.333-334 in [8])

$$\Delta_F = \exp_F \{ \partial_F \} - id.$$

Fib-Roman coefficients (as seen from the above) are then also natural "relative" of binomial coefficients among the family of ψ - binomial ones [8] (consult also Example 2.1 in [9]).

The Logarithmic Fib-Binomial Formula extends the notion of binomiality of polynomials as used in the Generalized Umbral Calculus (see Chapter 6 in [16] for functional formulation and see [8] for abundant references on Finite Operator Calculus of Rota formulation)- to sequences of functions - (compare with [1]).

Here the importance of the great invention of Steven Roman - among others - relies on the fact that the R -linear span L_F of now basis *Fib-harmonic logarithms* functions

$$\{ \phi_n^{(t)} \}_{n \in \mathbf{Z}, t=0,1}, \quad L_F = span \left| \{ \phi_n^{(t)} \}_{n \in \mathbf{Z}, t=0,1} \right|$$

allows the Fundamental Theorem of Calculus to hold also on L_F , i.e. $\partial_F^{-1} \partial_F = id_{L_F}$ for ∂_F - difference-ization and \int_F - integration acting on a linear space L_F being the linear span of "*Fib-harmonic logarithms*". Here anti-difference-ization operator ∂_F^{-1} - depending on whether $t = 0$ or $t = 1$ - acts as follows on *Fib-harmonic logarithm* functions :

$$\partial_F^{-1} = \int_0^x d_F \quad \text{on} \quad \phi_n^{(0)} \quad n \neq -1, \quad n \in \mathbf{Z} \quad \text{and gives } 0 \text{ for } n = -1$$

$$\text{and } \partial_F^{-1} = \int_1^x d_F \quad \text{on} \quad \phi_n^{(1)}; \quad n \in \mathbf{Z}$$

Let us define these *Fib-harmonic logarithms*

$$\{ \phi_n^{(t)} \}_{n \in \mathbf{Z}, t=0,1}$$

-(see Proposition 2.2 in [14]) - as solutions of *Fib-harmonic* t -binomiality conditions. Thus *Fib-harmonic logarithm functions* are unique solutions of *Fib-harmonic* t -binomiality conditions; $t = 0, 1$ (19) - (compare with [1] and relaxation Lemma 2.12 therein):

$$\begin{aligned} 1) \quad & \phi_0^{(0)}(x) = 1, \quad 2) \quad \phi_n^{(0)}(0) = 0, \quad n \ni \mathbf{Z} \setminus \{0\}, \\ 3) \quad & \partial_F \phi_n^{(0)} = [n_F] \phi_{n-1}^{(0)}, \quad n \in \mathbf{Z} \end{aligned} \tag{19}$$

$$1) \quad \phi_0^{(1)}(x) = \ln x, \quad 2) \quad \phi_n^{(1)}(x) \text{ has no constant term, } n \in \mathbf{Z},$$

$$3) \quad \partial_F \phi_n^{(1)} = [n_F] \phi_{n-1}^{(1)}, \quad n \in \mathbf{Z}$$

The *Fib-harmonic* t -binomiality conditions; $t = 0, 1$ (19) yield [14] what follows:

Proposition 4.1.

$$\begin{aligned} \phi_n^{(0)}(x) = \begin{cases} x^n & n \geq 0 \\ 0 & n < 0 \end{cases}, \quad \phi_n^{(1)}(x) = \begin{cases} x^n(\ln x - f_n) & n \geq 0 \\ x^n & n < 0 \end{cases}, \\ f_0 = 0, \quad f_n = 1 + \frac{1}{2_F} + \frac{1}{3_F} + \dots + \frac{1}{n_F}, \quad n \in \mathbf{N} \end{aligned}$$

We shall call f_n , $n \in \mathbf{N}$ the *Fib-harmonic numbers* ($f_0 = 0$), (see: [13]).

Proposition 4.2. *The linear anti-difference-ization unique operator $\partial_F^{-1} : L_F \longrightarrow L_F$; $\partial_F^{-1} \partial_F = id_{L_F}$ is given by*

$$\partial_F^{-1} \phi_n^{(0)} = \begin{cases} \frac{1}{[n+1]_F} \phi_{n+1}^{(0)} & n \neq -1 \\ 0 & n = -1 \end{cases}, \quad \partial_F^{-1} \phi_n^{(1)} = \frac{1}{[n+1]_F} \phi_n^{(1)}, \quad n \in \mathbf{Z}.$$

REMARK. Instead of Roman Fib-coefficients and Roman Fib-factorial one may - (replace F by ψ)- start to consider Roman ψ -coefficients, ψ -harmonic logarithms etc. However these seemingly might lack any "reasonable" combinatorial interpretation.

As the generally useful reading - also for this purpose one recommends here:

[LR] Loeb D., Rota G-C. "*Recent Contributions to the calculus of Finite Differences: a Survey*" Lecture Notes in Pure and Appl. Math. vol. 132(1991), pp. 239-276, ArXiv: math.CO/ 9502210 V1 9 Feb 1995, see also: <http://arxiv.org/list/math.CO/9502>

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